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$\mathbf{SL_q(2,\mathbb{R})}$ at roots of unity

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Abstract.

The quantum group $SL_q(2,\mathbb{R})$ at roots of unity is introduced by means of duality pairings with the quantum algebra $U_q(sl(2,\mathbb{R}))$. Its irreducible representations are constructed through the universal T–matrix. An invariant integral on this quantum group is given. Endowed with that some properties like unitarity and orthogonality of the irreducible representations are discussed.

1. Introduction

One of the most interesting features of the quantum algebra $U_q(sl(2))$ which does not possess classical analog is the finite dimensional cyclic irreducible representation which appears when q is a root of unity[1], [2]. Indeed, cyclic representations appear in different physical applications like generalized Potts model[3] and in classification of quantum Hall effect wave functions[4].

A geometric understanding of this feature is lacking due to the fact that structure of the related quantum group $SL_q(2)$ at roots of unity is not well established, although, there are encouraging results in this direction[5]–[6]. When q is not a root of unity $SL_q(2)$ and $U_q(sl(2))$ are duals of each other[7],[8]. Hence, it would be reasonable to extend this property to obtain $SL_q(2)$ when q is a root of unity. However, this is not straightforward, because when q is a root of unity the usual dual brackets become to be ill defined. To cure this shortcoming one should alter the usual number of variables taking part in the duality relations. Then one can define the quantum group $SL_q(2)$ at roots of unity. Obviously, this fact should be reflected in $U_q(sl(2))$ at roots of unity such that the number of variables needed to define it should be changed consistently.

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Our aim is to clarify construction of $SL_q(2)$ at roots of unity as dual of $U_q(sl(2))$ and study them in terms of the usual representation theory techniques. Because of the involutions adopted, indeed we work with $SL_q(2,\mathbb{R})$ and $U_q(sl(2,\mathbb{R}))$.

In the sequel we first discuss $U_q(sl(2,\mathbb{R}))$ and $SL_q(2,\mathbb{R})$ for a generic q in terms of some new variables which are suitable to define orthogonal duality pairings. Then, we discuss degeneracies arising in dual brackets when we deal with $q^p = 1$ for an odd integer p and present a procedure for getting rid of them. This yields the definition of $SL_q(2,\mathbb{R})$ at roots of unity, whose subgroups are also studied.

Once the concepts are clarified we first study irreducible representations of $U_q(sl(2,\mathbb{R}))$ and then work out the universal T–matrix. These representations as well as the T–matrix are utilized to find out irreducible representations of $SL_q(2,\mathbb{R})$ at roots of unity. Finally, we give the definition of invariant integral on $SL_q(2,\mathbb{R})$ and discuss the related structure of the representations like unitarity and orthogonality.

2. $\mathbf{U_q(sl(2,\mathbb{R}))}$ and $\mathbf{SL_q(2,\mathbb{R})}$ for a generic $\mathbf{q}$

The quantum algebra $U_q(sl(2,\mathbb{R}))$ is the $*$–Hopf algebra generated by $E_\pm$ and $K^{\pm 1}$ which satisfy the commutation relations

$$KE_\pm K^{-1} = q^{\pm 1} E_\pm,\ [E_+, E_-] = \frac{K^2 - K^{-2}}{q - q^{-1}}, \tag{2.1}$$

the comultiplications

$$\Delta(E_\pm) = E_\pm \otimes K + K^{-1} \otimes E_\pm, \qquad \Delta(K) = K \otimes K, \tag{2.2}$$

the counits, the antipodes

$$\epsilon(K) = 1, \qquad \epsilon(E_\pm) = 0, \tag{2.3}$$
$$S(K) = K^{-1}, \qquad S(E_\pm) = -q^{\mp 1} E_\pm \tag{2.4}$$

and the involutions

$$E_\pm^* = E_\pm,\ K^* = K. \tag{2.5}$$

The quantum group $SL_q(2,\mathbb{R})$ is the $*$–Hopf algebra $A(SL_q(2,\mathbb{R}))$ generated by x, y, u and v satisfying the commutation relations

$$\begin{aligned} &ux = qxu, \quad vx = qxv, \quad yu = quy, \\ &yv = qvy, \quad uv = vu, \quad yx - quv = xy - q^{-1}uv = 1, \end{aligned} \tag{2.6}$$

the comultiplications

$$\begin{aligned} \Delta x = x \otimes x + u \otimes v \quad &, \quad \Delta u = x \otimes u + u \otimes y, \\ \Delta v = v \otimes x + y \otimes v \quad &, \quad \Delta y = v \otimes u + y \otimes y, \end{aligned} \tag{2.7}$$

the counits, the antipodes

$$\epsilon(x) = 1, \qquad \epsilon(y) = 1, \qquad \epsilon(u) = 0, \qquad \epsilon(v) = 0, \tag{2.8}$$
$$S(x) = y, \qquad S(y) = x, \qquad S(u) = -qu, \qquad S(v) = -q^{-1}v \tag{2.9}$$

and the involutions

$$x^* = x,\ y^* = y,\ u^* = u,\ v^* = v. \tag{2.10}$$

The involutions adopted (2.10) and the Hopf algebra operations (2.6)–(2.9) imply $|q| = 1$.

Assume that there exists a $*$–representation of $A(SL_q(2,\mathbb{R}))$ such that x admits the inverse x^{-1} and the equality

$$(1_A + q^{-1}uv)^{-1} = \sum_{k=0}^{\infty}(-1)^k(q^{-1}uv)^k \tag{2.11}$$

holds. 1_A and 1_U indicate the unit elements of the related Hopf algebras.

Then, introduce the new variables

$$\eta_+ = q^{-1/2}ux;\ \eta_- = q^{1/2}vx^{-1};\ \delta = x^2, \tag{2.12}$$

dictated by the Gauss decomposition

$$\left(\begin{array}{cc} x & u \\ v & y \end{array}\right) = \left(\begin{array}{cc} 1_A & 0 \\ q^{-1/2}\eta_- & 1_A \end{array}\right)\left(\begin{array}{cc} 1_A & q^{1/2}\eta_+ \\ 0 & 1_A \end{array}\right)\left(\begin{array}{cc} \delta^{1/2} & 0 \\ 0 & \delta^{-1/2} \end{array}\right), \tag{2.13}$$

satisfying the commutation relations

$$\eta_-\eta_+ = q^2\eta_+\eta_-, \qquad \eta_\pm\delta = q^2\delta\eta_\pm. \tag{2.14}$$

The involutions (2.10) yield

$$\eta_\pm^* = \eta_\pm;\ \delta^* = \delta. \tag{2.15}$$

Through the equality (2.11) we can define the following Hopf algebra operations on these variables:

$$\Delta\delta = \delta\otimes\delta + q^{-2}\delta^{-1}\eta_+^2\otimes\eta_-^2\delta + (1_A + q^{-2})\eta_+\otimes\eta_-\delta, \tag{2.16}$$
$$\Delta\eta_+ = \eta_+\otimes 1_A + \delta\otimes\eta_+ + (1_A + q^2)\eta_+\otimes\eta_+\eta_- + q^{-2}\delta^{-1}\eta_+^2\otimes(1_A + q^2\eta_+\eta_-)\eta_-, \tag{2.17}$$
$$\Delta\eta_- = \eta_-\otimes 1_A + \delta^{-1}\otimes\eta_- + \sum_{k=1}^{\infty}(-1)^k q^{-k(k+1)}\delta^{-k-1}\eta_+^k\otimes\eta_-^{k+1}, \tag{2.18}$$
$$S(\delta) = \delta^{-1}(1_A + q^{-2}\eta_+\eta_-)(1_A + \eta_+\eta_-), \quad S(\eta_\pm) = -\delta^{\mp 1}\eta_\pm, \tag{2.19}$$
$$\epsilon(\delta) = 1, \quad \epsilon(\eta_\pm) = 0. \tag{2.20}$$

When q is not a root of unity duality relations between $U_q(sl(2,\mathbb{R}))$ and $A(SL_q(2,\mathbb{R}))$ are given by

$$\langle K^i\ ,\ \delta^j\rangle = q^{ij}, \qquad\qquad i,j\in\mathbb{Z}, \tag{2.21}$$

$$\langle E_\pm^n\ ,\ \eta_\pm^m\rangle = i^n q^{\pm n/2}[n]!\delta_{n,m}, \qquad\qquad n,m\in\mathbb{N}, \tag{2.22}$$

where

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

is the q–number.

3. $\mathbf{U_q(sl(2,\mathbb{R}))}$ and $\mathbf{SL_q(2,\mathbb{R})}$ when $\mathbf{q^p = 1}$

When $q^p = 1$ (we deal with p =odd integer) for any integer j we have the conditions $q^{jp} = 1$ and $[jp] = 0$. So that, the dual brackets (2.21) and (2.22) are degenerate. To remove the degeneracy in (2.21) we put the restrictions

$$K^p = 1_U, \qquad\qquad \delta^p = 1_A. \tag{3.1}$$

By means of these conditions and the new variables

$$\mathcal{D}(m) \equiv \frac{1}{p}\sum_{l=0}^{p-1} q^{-lm}\delta^l,$$

instead of (2.21) we have

$$\langle K^n\ ,\ \mathcal{D}(m)\rangle = \delta_{n,m},\ \ n,m\in[0,p-1]. \tag{3.2}$$

Removing the degeneracies in (2.22) can be achieved in terms of the following two procedures. Take $m,n\in[0,p-1]$ in (2.22). Let

$$\eta_\pm^p = 0, \tag{3.3}$$

but introduce the new variables

$$z_\pm \equiv \lim_{q^p=1}\frac{\eta_\pm^p}{[p]!}, \tag{3.4}$$

without any condition on $E_\pm$. In the second procedure there is no condition on $\eta_\pm$ but on the generators of $U_q(sl(2,\mathbb{R}))$: $E_\pm^p = 0$ with the new variables $Z_\pm \equiv \lim_{q^p=1}\frac{E_\pm^p}{[p]!}$. Existence of these limits $z_\pm$ and $Z_\pm$ is discussed in ([2], [9] and references therein).

Although, there is one more way of defining new variables by setting both $E_\pm^p = 0$ and $\eta_\pm^p = 0$ which is studied in [6], we will show that it can be obtained as a special case in our approach.

We deal with the restrictions (3.3) and the new variables (3.4). Now, the duality relations are

$$\langle E_\pm^n\ ,\ \eta_\pm^m\rangle = i^n q^{\pm n/2}[n]!\delta_{n,m}, \quad n,m\in[0,p-1] \tag{3.5}$$

and

$$\langle \mathcal{E}_\pm^s\ ,\ z_\pm^t\rangle = i^s s!\delta_{s,t}, \qquad\qquad s,t\in\mathbb{N}, \tag{3.6}$$

where $\mathcal{E}_\pm \equiv (-1)^{\frac{p+1}{2}}E_\pm^p$. Obviously, $z_\pm$ commute with the other elements and satisfy the Hopf algebra operations

$$S(z_\pm) = -z_\pm, \qquad\qquad \epsilon(z_\pm)=0, \qquad\qquad z_\pm^* = z_\pm, \tag{3.7}$$

$$\Delta z_+ = z_+\otimes 1_A + 1_A\otimes z_+ + \sum_{k=1}^{p-1}\frac{q^{k^2}}{[k]![p-k]!}\eta_+^{p-k}\delta^k\otimes(-q^2\eta_+\eta_-;q^2)_{(p-k)}\eta_+^k, \tag{3.8}$$

$$\Delta z_- = z_-\otimes 1_A + 1_A\otimes z_- + \sum_{k=1}^{p-1}\frac{q^{-k^2}}{[k]![p-k]!}\eta_-^{p-k}\delta^{-k}(-\eta_+\eta_-;q^{-2})_k\otimes\eta_-^k, \tag{3.9}$$

where we used the notation

$$(a;q)_k \equiv \prod_{j=1}^k (1-aq^{j-1}).$$

Let, $SL_q(2,\mathbb{R}|p)$ denotes the $*$–Hopf algebra $A(SL_q(2,\mathbb{R}|p))$ generated by $\eta_\pm$ and δ through the Hopf structure given by (2.14)–(2.20). Due to the restrictions (3.1) and (3.3) $SL_q(2,\mathbb{R}|p)$ is a finite group with dimension p^3.

When we deal with any $f(z_+,z_-)\equiv f(z)\in C^\infty(\mathbb{R}^2)$ (the space of all infinitely differentiable functions on $\mathbb{R}^2$)

$$\Delta(f(z)) = f(z_0) + f'_+(z_0)c_+ + f'_-(z_0)c_- + f''_{+-}(z_0)c_+c_-, \tag{3.10}$$

where $z_0\equiv(z_+\otimes 1_A + 1_A\otimes z_+, z_-\otimes 1_A + 1_A\otimes z_-)$ and $c_\pm$ are given by the remaining terms of (3.8), (3.9) which are nilpotent $c_\pm^2=0$. Here, $f'_\pm(z_0)$ and $f''_{+-}(z_0)$ indicate derivatives of f with respect to $z_\pm$ and z_+z_- evaluated at z_0. We also have

$$S(f(z)) = f(-z), \quad \epsilon(f(z)) = f(0), \quad f(z)^* = \overline{f(z)}, \tag{3.11}$$

where bar indicates complex conjugation.

We are ready to give the definition:

$SL_q(2,\mathbb{R})$ at roots of unity ($q^p=1$) is the $*$–algebra $A(SL_q(2,\mathbb{R})) = A(SL_q(2,\mathbb{R}|p))\times C^\infty(\mathbb{R}^2)$ possessing the Hopf algebra structure given by (2.14)–(2.20) and (3.10)–(3.11).

Let the convolution product $\xi : A \to V$ be a homomorphic map of the Hopf algebra A onto the linear space V. We set

$$\xi \diamond g = (id \otimes \xi)\Delta(g), \quad g \diamond \xi = (\xi \otimes id)\Delta(g), \quad \xi \diamond \xi = (\xi \otimes \xi)\Delta. \tag{3.12}$$

$\xi \diamond g$ and $g \diamond \xi$ belong to $A \otimes V$ and $V \otimes A$, respectively; $\xi \diamond \xi$ is a homomorphic map of $A \otimes A$ onto $V \otimes V$.

Obviously, $SL_q(2,\mathbb{R}|p)$ is an invariant subgroup of $SL_q(2,\mathbb{R})$ at roots of unity. Moreover, in terms of the homomorphisim $\xi_c : A(SL_q(2,\mathbb{R})) \to C^\infty(\mathbb{R}^2)$:

$$\xi_c(\eta_\pm) = 0, \quad \xi_c(\delta) = 1, \quad \xi_c(z_\pm) = z_\pm, \tag{3.13}$$

one can observe that the comultiplication (3.10) yields

$$\xi_c \diamond \xi_c(f(z)) = f(z_0). \tag{3.14}$$

Written on the coordinates $z_\pm$:

$$\xi_c \diamond \xi_c(z_\pm) = z_\pm \otimes 1_A + 1_A \otimes z_\pm, \tag{3.15}$$

indicates that $*$–Hopf algebra $C^\infty(\mathbb{R}^2)$ is the translation group which is a subgroup of the $SL_q(2,\mathbb{R})$ at roots of unity.

There is another subgroup $SO(1,1|p)$, given in terms of the homomorphism

$$\xi_t(\eta_\pm) = 0, \quad \xi_t(\delta) = t, \tag{3.16}$$

where $t^p = 1$. The right sided coset $\mathcal{C}_q^{(1,1)} = SL_q(2,\mathbb{R}|p)/SO(1,1|p)$ is the subspace of $A(SL_q(2,\mathbb{R}|p))$ defined by

$$A(\mathcal{C}_q^{(1,1)}) = \{g \in A(SL_q(2,\mathbb{R}|p)) : \quad \xi_t \diamond g = g \otimes 1_A\}. \tag{3.17}$$

One can show that

$$\xi_t \diamond \eta_+^n \eta_-^m \delta^k = \eta_+^n \eta_-^m \delta^k \otimes t^k. \tag{3.18}$$

So that, $\eta_+^n \eta_-^m$, n, $m \in [0, p-1]$ form a basis of $A(\mathcal{C}_q^{(1,1)})$. Observe that

$$e_{nm}^\pm = \frac{\eta_+^{p-1-n}\eta_-^{p-1-m} \pm \eta_+^n \eta_-^m}{\sqrt{q^{2n+1} + q^{-2n-1}}}, \qquad n, m \in [0, p-1] \tag{3.19}$$

define a basis which are independent in the range

$$n \in [0, n_0 - 1], \quad m \in [0, 2n_0]; \qquad n = n_0, \quad m \in [0, n_0], \tag{3.20}$$

where $n_0 = \frac{p-1}{2}$. The number of independent elements of e^+_{nm} and e^-_{nm} are $\frac{p^2+1}{2}$ and $\frac{p^2-1}{2}$. The quantum hyperboloid $H_q^{(1,1)} = SL_q(2,\mathbb{R})/SO(1,1|p)$ is defined through the subspace of $A(SL_q(2,\mathbb{R}))$

$$A(H_q^{(1,1)}) = A(\mathcal{C}_q^{(1,1)}) \times C^\infty(\mathbb{R}^2). \tag{3.21}$$

The homomorphism

$$\xi_l(\eta_+) = \eta, \qquad \xi_l(\eta_-) = 0, \qquad \xi_l(\delta) = t, \tag{3.22}$$

defines another subgroup of $SL_q(2,\mathbb{R})$ denoted by $E_q(1)$. Its Hopf algebra structure is inherited from that of $A(SL_q(2,\mathbb{R}))$. The right sided coset $\mathbb{R}_q = SL_q(2,\mathbb{R}|p)/E_q(1)$ is given through the subspace

$$A(\mathbb{R}_q) = \{g \in A(SL_q(2|p)) : \quad \xi_l \diamond g = g \otimes 1_A\}. \tag{3.23}$$

Observe that elements of this space are polynomials in η_-.

We should also define:

The quantum algebra $U_q(sl(2,\mathbb{R}))$ at roots of unity is generated by $E_\pm$, $\mathcal{E}_\pm$ and K with the restriction $K^p = 1_U$. Its basis elements are

$$\mathcal{E}_+^s \mathcal{E}_-^t E_+^m E_-^n K^k \qquad n,m,k \in [0,p-1], \qquad s,t \in \mathbb{N}.$$

Its $*$–Hopf algebra structure is given by (2.1)–(2.5) and

$$\Delta(\mathcal{E}_\pm) = \mathcal{E}_\pm \otimes 1_U + 1_U \otimes \mathcal{E}_\pm, \quad S(\mathcal{E}_\pm) = -\mathcal{E}_\pm, \quad \epsilon(\mathcal{E}_\pm) = 0, \quad \mathcal{E}_\pm^* = \mathcal{E}_\pm.$$

In terms of the homomorphism $\xi_a : U_q(sl(2,\mathbb{R})) \to U_q(sl(2,\mathbb{R}|p))$

$$\xi_a(E_\pm) = E_\pm, \qquad \xi_a(K) = K, \qquad \xi_a(\mathcal{E}_\pm) = 0,$$

we can define $U_q(sl(2,\mathbb{R}|p))$ the sub–Hopf algebra of $U_q(sl(2,\mathbb{R}))$ generated by

$$E_\pm^p = 0, \qquad K^p = 1_U.$$

Obviously, the discrete quantum algebra $U_q(sl(2,\mathbb{R}|p))$ is in non–degenerate duality with $SL_q(2,\mathbb{R}|p)$. This is the case studied in [6].

4. Irreducible $*$-representations of $\mathbf{U_q(sl(2,\mathbb{R}))}$ when $\mathbf{q^p = 1}$

The homomorphism $\mathcal{L}^\lambda :\ U_q(sl(2)) \to \mathrm{Lin}\ A(SO(1,1|p))$ given by

$$\begin{aligned}
\mathcal{L}^\lambda(K)t^i &= q^{-i}t^i, \quad i \in [0,p-1],\\
\mathcal{L}^\lambda(E_-)t^i &= t^{i+1}, \quad i = 0,1,\cdots,p-2,\\
\mathcal{L}^\lambda(E_-)t^{p-1} &= \lambda_+ t^0,\\
\mathcal{L}^\lambda(E_+)t^i &= M_i t^{i-1} \ \ i = 1,\cdots,p-1,\\
\mathcal{L}^\lambda(E_+)t^0 &= a t^{p-1} \quad ,
\end{aligned} \tag{4.1}$$

where the constants are

$$\lambda_- = a\prod_{i=1}^{p-1} M_i,\;\; M_i = a\lambda_+ - [i-1][i],$$

defines the cyclic irreducible representation of $U_q(sl(2))$ ($\mathcal{B}$ type representation)[1],[2].

We would like to find out when $\mathcal{L}^\lambda$ defines a $*$–representation. To this aim introduce the Hermitian form

$$(a,b)_t = \mathcal{I}_t(a^* b), \tag{4.2}$$

for $a, b \in A(SO(1,1|p))$ and the linear functional on it

$$\mathcal{I}_t(t^m) = \delta_{m,0(\mathrm{mod}\;\; p)}. \tag{4.3}$$

Moreover, we see that

$$e_m^{\pm} = \frac{1}{\sqrt{2}}(t^m \pm t^{p-m}), \qquad\qquad m \in [0, \frac{p-1}{2}],$$

are orthogonal with respect to the Hermitian form (4.2):

$$(e_m^{\pm}, e_k^{\pm})_t = \pm\delta_{mk}, \qquad\qquad (e_m^{\mp}, e_k^{\pm})_t = 0.$$

Thus, with the Hermitian form (4.2) $*$–Hopf algebra $A(SO(1,1|p))$ is the pseudo–Euclidean space possessing $\frac{p+1}{2}$ positive and $\frac{p-1}{2}$ negative signatures.

Adjoint of a linear operator is defined through

$$(\mathcal{L}^\lambda(\phi)a, b)_t = (a, (\mathcal{L}^\lambda(\phi))^* b)_t,$$

where $\phi \in U_q(sl(2,\mathbb{R}))$. Hence, we conclude that if $\lambda_\pm$ are real $\mathcal{L}^\lambda$ defines a $*$–representation:

$$(\mathcal{L}^\lambda(\phi))^* = \mathcal{L}^\lambda(\phi^*).$$

The linear map $T^{(l)}\colon\; A(\mathbb{R}_q) \to A(SL_q(2,\mathbb{R})) \times A(\mathbb{R}_q)$ given by

$$T^{(l)} g(\eta_-) = \left(id \otimes \delta^{-l}\right)\Delta(\delta^l g(\eta_-)),$$

for $l \in [0, \frac{p-1}{2}]$ defines irreducible representations of $SL_q(2,\mathbb{R})$. Infinitesimal form of this global representation is

$$\mathcal{R}^{(l)}(\phi) g(\eta_-) = (\phi \otimes id) T^{(l)} g(\eta_-),$$

where $\phi \in U_q(sl(2,\mathbb{R}))$. We see that

$$\begin{aligned}
\mathcal{R}^{(l)}(E_+)\eta_-^{l-m} &= i q^{l+1/2}[l+m]\eta_-^{l-m+1},\\
\mathcal{R}^{(l)}(E_-)\eta_-^{l-m} &= i q^{-l-1/2}[l-m]\eta_-^{l-m-1},\\
\mathcal{R}^{(l)}(K)\eta_-^{l-m} &= q^m \eta_-^{l-m},
\end{aligned}$$

where $m \in [-l, l]$. These are non–cyclic representations of $U_q(sl(2,\mathbb{R}))$ ($\mathcal{A}$ type representations).

5. The universal T–matrix and irreducible representations of $\mathbf{SL_q(2,\mathbb{R})}$ at roots of unity

Let the basis elements of the Hopf algebras $U(g)$ and $A(G)$, respectively, V_a and v^a lead to the dual brackets $\langle V_a, v^b\rangle = \delta_a^b$, which are non–degenerate. Then the universal T–matrix $T \in U(g) \otimes A(G)$ can be constructed as[10],[11]

$$T = \sum_a V_a \otimes v^a.$$

As far as the universal T–matrix is known, one can construct corepresentations of $A(G)$ utilizing representations of $U(g)$.

A straightforward calculation leads to the duality brackets

$$\begin{aligned}\langle \mathcal{E}_+^t \mathcal{E}_-^s E_+^n E_-^m K^k \ ,\ z_+^{t'} z_-^{s'} \eta_+^{n'} \eta_-^{m'} \mathcal{D}(k')\rangle =& i^{s+t+n+m} q^{\frac{(n-m)}{2}-nm} s!t![m]![n]! \\ & \delta_{n,n'}\delta_{m,m'}\delta_{s,s'}\delta_{t,t'}\delta_{k+n+m,k'},\end{aligned} \tag{5.1}$$

where $n, m \in [0, p-1]$. Therefore, the universal T–matrix can be written as

$$T = e^{-i\mathcal{E}_+\otimes z_+ - i\mathcal{E}_-\otimes z_-} \sum_{n,m,k=0}^{p-1} \frac{i^{-n-m} q^{\frac{m-n}{2}+nm}}{[n]![m]!} E_+^n E_-^m K^k \otimes \eta_+^n \eta_-^m \mathcal{D}(k+n+m). \tag{5.2}$$

Arranging the elements and using the cut off q–exponentials

$$e_\pm^x = \sum_{r=1}^{p-1} \frac{q^{\pm r(r-1)/2}}{[r]!} x^r,$$

the universal T–matrix can also be written as

$$T = e^{-i\mathcal{E}_+\otimes z_+ - i\mathcal{E}_-\otimes z_-} e_+^{i\epsilon_+\otimes\eta_+} e_-^{i\epsilon_-\otimes\eta_-} D(K,\delta), \tag{5.3}$$

where we introduced

$$\begin{aligned}\epsilon_\pm &= -q^{\pm 1/2} E_\pm K^{-1}, \\ D(K,\delta) &= \frac{1}{p} \sum_{k,l=0}^{p-1} q^{-ml} K^k \otimes \delta^l.\end{aligned}$$

Using the explicit form (5.3) one can show that

$$[(*\otimes *)T] \cdot T = 1_A \otimes 1_U, \quad T \cdot (*\otimes *)T = 1_A \otimes 1_U. \tag{5.4}$$

In general, T–matrix also satisfies

$$(id \otimes \Delta)T = (T \otimes 1_A)(id \otimes \sigma)(T \otimes 1_A), \tag{5.5}$$

where $\sigma(F \otimes G) = G \otimes F$, $F,\ G \in A(SL_q(2,\mathbb{R}))$, is the permutation operator.

Let us illustrate how one obtains irreducible representations of $SL_q(2,\mathbb{R})$ by making use of the universal T–matrix (5.2). Let $T^{(\lambda)} : A(SO(1,1|p)) \to A(SO(1,1|p)) \otimes A(SL_q(2,\mathbb{R}))$, be

$$T^{(\lambda)}a = e^{-i\mathcal{L}^\lambda(\mathcal{E}_+)\otimes z_+ - i\mathcal{L}^\lambda(\mathcal{E}_-)\otimes z_-} e_+^{i\mathcal{L}^\lambda(\epsilon_+)\otimes\eta_+} e_-^{i\mathcal{L}^\lambda(\epsilon_-)\otimes\eta_-} D(\mathcal{L}^\lambda(K),\delta). \tag{5.6}$$

Because of (5.5) and the irreducibility of the representation $\mathcal{L}^\lambda$ we conclude that $T^{(\lambda)}a$ gives a p–dimensional irreducible representation of the quantum group $SL_q(2,\mathbb{R})$ in the linear space $A(SO(1,1|p))$. Let us extend the Hermitian form (4.2) to

$$\{a \otimes F, b \otimes G\}_t = (a,b)_t F^* G, \tag{5.7}$$

where $F,\ G \in A(SL_q(2,\mathbb{R}))$ and $a,\ b \in A(SO(1,1|p))$. When $\lambda_\pm$ are real numbers the condition (5.4) yields

$$\{T^{(\lambda)}a, T^{(\lambda)}b\}_t = (a,b)_t 1_A. \tag{5.8}$$

Thus the irreducible representation $T^{(\lambda)}$ is pseudo–unitary when $\lambda_\pm$ are real.

We can obtain matrix elements of the irreducible pseudo–unitary representations as

$$D^\lambda_{mn} = \{t^{p-m} \otimes 1_A, T^{(\lambda)}t^n\}_t. \tag{5.9}$$

For some specific values of $n,\ m$ we performed the explicit calculations:

$$D^\lambda_{00} = e^{-i\lambda_+ z_+ - i\lambda_- z_-}\{1 + \sum_{m=1}^{p-1} \frac{(-1)^m}{([m]!)^2} \left(\prod_{j=1}^{m} M_j\right) \rho^m\}, \tag{5.10}$$

where $\rho = q\eta_+\eta_-$. For $i \neq 0$, we obtain

$$\begin{aligned} D^\lambda_{i0} \;\; = \;\; & e^{-i\lambda_+ z_+ - i\lambda_- z_-}\{\sum_{m=0}^{p-i-1} \frac{(-1)^m i^{-i} q^{i(m-1/2)}}{[m]![m+i]!} \left(\prod_{j=1}^{m+i} M_j\right) \rho^m \eta_-^i \\ & + \sum_{m=0}^{i-1} \frac{(-1)^m i^{i-p} q^{i(p-1)/2-im}}{[m]![p+m-i]!} \left(\prod_{j=0}^{m} M_j\right) \eta_+^{p-i} \rho^m\}, \end{aligned} \tag{5.11}$$

where the definition $M_0 \equiv \lambda_+$ is adopted.

The pseudo–unitarity condition (5.8) implies

$$(D^\lambda_{0m})^* D^\lambda_{0n} + \sum_{k=1}^{p-1} (D^\lambda_{km})^* D^\lambda_{p-kn} = (t^m, t^n)_t 1_A. \tag{5.12}$$

Special cases are

$$(D^{\lambda}_{00})^* D^{\lambda}_{00} + \sum_{k=1}^{p-1} (D^{\lambda}_{0k})^* D^{\lambda}_{0p-k} = 1_A,$$
$$(D^{\lambda}_{0i})^* D^{\lambda}_{0p-i} + \sum_{k=1}^{p-1} (D^{\lambda}_{ki})^* D^{\lambda}_{p-kp-i} = 1_A. \tag{5.13}$$

Moreover, we have the addition theorem

$$\Delta(D^{\lambda}_{nm}) = \sum_{k=0}^{p-1} D^{\lambda}_{nk} \otimes D^{\lambda}_{km}.$$

6. Regular representation of $\mathbf{SL_q(2,\mathbb{R})}$

The comultiplication

$$\Delta : A(SL_q(2,\mathbb{R})) \to A(SL_q(2,\mathbb{R})) \otimes A(SL_q(2,\mathbb{R})) \tag{6.1}$$

defines the regular representation of $SL_q(2,\mathbb{R})$ in the linear space $A\left(SL_q(2,\mathbb{R})\right)$. The right and left representations of $U_q(sl(2,\mathbb{R}))$ corresponding to the regular representation (6.1) are given, respectively, by

$$\mathcal{R}(\phi)F \equiv \hat{\phi}F = F \diamond \phi$$

and

$$\mathcal{L}(\phi)F \equiv \tilde{\phi}F = \phi \diamond F,$$

where $F \in A\left(SL_q(2,\mathbb{R})\right)$. Straightforward calculations yield the right representations

$$\begin{aligned}
\hat{E}_+\eta_+^n &= iq^{1/2}[n]\eta_+^{n-1} + iq^{1/2-n}[2n]\eta_-\eta_+^n, & \hat{E}_+\eta_-^n &= -iq^{1/2}[n]\eta_-^{n+1},\\
\hat{E}_-\eta_-^n &= iq^{-1/2}[n]\eta_-^{n-1}, & \hat{K}\eta_\pm^n &= q^{\pm n}\eta_\pm^n,\\
\hat{E}_-\eta_+^n &= 0, & \hat{E}_-\delta^n &= 0,\\
\hat{E}_+\delta^n &= i(q^{-3/2-n} + q^{-3n-7/2})[n+1]\eta_-\delta^n(1-\delta_{n,0}), & \hat{K}\delta^n &= q^n\delta^n,\\
\hat{E}_\pm f(z_+,z_-) &= \frac{iq^{\pm 1/2}}{[p-1]!}\eta_\pm^{p-1}\frac{df(z_+,z_-)}{dz_\pm}, & \hat{K}z_\pm &= z_\pm,
\end{aligned}$$

and the left representations

$$\begin{aligned}
\tilde{E}_+\eta_+^n &= iq^{n-3/2}[n]\delta\eta_+^{n-1}, & \tilde{E}_-\eta_+^n &= iq^{-n-1/2}\delta^{-1}\eta_+^{n+1},\\
\tilde{E}_-\eta_-^n &= iq^{3/2-n}[n]\delta^{-1}\eta_-^{n-1}, & \tilde{K}\eta_\pm^n &= \eta_\pm^n,\\
\tilde{E}_+\eta_-^n &= 0, & \tilde{E}_+\delta^n &= 0,\\
\tilde{E}_-\delta^n &= iq^{3/2-n}[2n]\eta_+\delta^{n-1}(\delta_{n,0}-1), & \tilde{K}\delta^n &= q^n\delta^n,\\
\tilde{E}_\pm f(z_+,z_-) &= \frac{iq^{\mp 1}}{[p-1]!}\eta_\pm^{p-1}\delta^{\pm}\frac{df(z_+,z_-)}{dz_\pm}, & \tilde{K}z_\pm &= z_\pm.
\end{aligned}$$

Right representation of any element $\phi \in U_q(sl(2,\mathbb{R}))$ can be found through the above relations and making use of the properties

$$\begin{aligned}
\mathcal{R}(\phi\phi') &= \mathcal{R}(\phi')\mathcal{R}(\phi),\\
\hat{E}_\pm(XY) &= \hat{E}_\pm X\hat{K}Y + \hat{K}^{-1}X\hat{E}_\pm Y,\\
\hat{K}XY &= \hat{K}X\hat{K}Y.
\end{aligned}$$

For left representations similar properties hold.

Although, the quantum algebra $U_q(sl(2,\mathbb{R}))$ at roots of unity possesses three Casimir elements $\mathcal{E}_\pm$ and

$$C = E_-E_+ + \frac{(qK - q^{-1}K^{-1})^2}{(q^2 - q^{-2})^2},$$

only two of them are independent. Thus, irreducible representations of $U_q(sl(2,\mathbb{R}))$ at roots of unity are labeled by two indices. A method of constructing the irreducible representations of $U_q(sl(2,\mathbb{R}))$ at roots of unity is to diagonalize the complete set of commuting operators $\hat{\mathcal{E}}_\pm$, $\hat{C}$ and $\hat{K}$ on the quantum hyperboloid. Indeed, the matrices (5.10) and (5.11) can be shown to satisfy

$$\begin{aligned}
\hat{C}D_{i0} &= a\lambda_+ D_{i0}, && i \in [0, p-1],\\
\hat{\mathcal{E}}_\pm D_{i0} &= (-1)^{\frac{p+1}{2}}\lambda_\pm D_{i0}, && i \in [0, p-1],\\
\hat{E}_+ D^\lambda_{i0} &= D^\lambda_{(i+1)0}, && i \in [0, p-2],\\
\hat{E}_+ D^\lambda_{(p-1)0} &= \lambda_+ D^\lambda_{0,0},\\
\hat{E}_- D^\lambda_{i0} &= M_i D^\lambda_{(i-1)0}, && i \in [1, p-1],\\
\hat{E}_- D^\lambda_{00} &= a D^\lambda_{0(p-1)}.
\end{aligned}$$

Similar constructions can also be done in terms of the left representations.

7. Invariant integral on $\mathbf{SL_q(2,\mathbb{R})}$ at roots of unity

Recall that the invariant integral $\mathcal{I}$ on the quantum group G_q is a linear functional on the Hopf algebra $A(G_q)$ which for any element $a \in A(G_q)$ satisfies the left

$$\mathcal{I} \diamond a = 1_A \mathcal{I}(a) \tag{7.1}$$

and the right

$$a \diamond \mathcal{I} = 1_A \mathcal{I}(a) \tag{7.2}$$

invariance conditions.

The linear functional $\mathcal{I}_p$ on the Hopf algebra $A(SL_q(2,\mathbb{R}|p))$ given by

$$\mathcal{I}_p(\eta_+^n\eta_-^m\delta^k) = q^{-1}\delta_{n,p-1}\delta_{m,p-1}\delta_{k,0(\mathrm{mod}\ p)} \tag{7.3}$$

defines the invariant integral on the quantum group $SL_q(2,\mathbb{R}|p)$. To prove that in fact the conditions (7.1) and (7.2) are satisfied, we proceed as follows. Since $A(SL_q(2,\mathbb{R}|p))$ is a finite Hopf algebra it is sufficient to show that (7.1) and (7.2) are satisfied after taking their dual pairings:

$$\begin{aligned} \mathcal{I}_p\left(\mathcal{R}(\phi)P\right) &= \mathcal{I}_p(P)\epsilon(\phi), & (7.4)\\ \mathcal{I}_p\left(\mathcal{L}(\phi)P\right) &= \mathcal{I}_p(P)\epsilon(\phi), & (7.5)\end{aligned}$$

for all elements $\phi \in U_q(sl(2,\mathbb{R}))$ and $P \in A(SL_q(2,\mathbb{R}|p))$. One can show that

$$\begin{aligned} &\mathcal{I}_p(\hat{E}_\pm\eta_+^n\eta_-^m\delta^k) = 0, \qquad \mathcal{I}_p(\hat{K}\eta_+^n\eta_-^m\delta^k) = \mathcal{I}_p(\eta_+^n\eta_-^m\delta^k), & (7.6)\\ &\mathcal{I}_p(\tilde{E}_\pm\eta_+^n\eta_-^m\delta^k) = 0, \qquad \mathcal{I}_p(\tilde{K}\eta_+^n\eta_-^m\delta^k) = \mathcal{I}_p(\eta_+^n\eta_-^m\delta^k). & (7.7)\end{aligned}$$

Moreover, for any two elements $\phi_1,\ \phi_2$ right and left representation satisfy the relations

$$\begin{aligned} \mathcal{I}_p(\mathcal{R}(\phi_1\phi_2)P) &= \epsilon(\phi_1\phi_2)\mathcal{I}_p(P),\\ \mathcal{I}_p(\mathcal{L}(\phi_1\phi_2)P) &= \epsilon(\phi_1\phi_2)\mathcal{I}_p(P).\end{aligned}$$

Therefore, (7.4) and (7.5) are satisfied. This leads to the conclusion that (7.3) is the invariant integral on $SL_q(2,\mathbb{R}|p)$.

Observe that

$$\mathcal{I}_p(P^*) = \overline{\mathcal{I}_p(P)} \tag{7.8}$$

and define the Hermitian form $(\cdot,\cdot)_p$ on the quantum group $SL_q(2,\mathbb{R}|p)$ as

$$(P,Q)_p = \mathcal{I}_p(PQ^*). \tag{7.9}$$

The basis elements $e_{nm}^\pm$ (3.19) of $A(\mathcal{C}_q^{(1,1)})$ are orthonormal in terms the above form:

$$(e_{nm}^\pm, e_{n'm'}^\pm)_p = \pm\delta_{nn'}\delta_{mm'}, \quad (e_{nm}^\pm, e_{n'm'}^\mp)_p = 0.$$

Any element $\pi \in A(\mathcal{C}_q^{(1,1)})$ can be represented as

$$\pi = \sum_{nm}\pi_{nm}^+e_{nm}^+ + \sum_{nm}\pi_{nm}^-e_{nm}^-, \tag{7.10}$$

where $\pi_{nm}^\pm \in \mathbb{C}$ and n, m take values in the domain (3.20). Then, the pseudo–norm of π

$$(\pi,\pi)_p = \sum_{nm}\pi_{nm}^+\overline{\pi_{nm}^+} - \sum_{nm}\pi_{nm}^-\overline{\pi_{nm}^-}, \tag{7.11}$$

shows that the metric of the space $A(\mathcal{C}_q^{(1,1)})$ possesses $\frac{p^2+1}{2}$ positive and $\frac{p^2-1}{2}$ negative signatures.

We should also define invariant integral on the translation subgroup for being able to obtain it on $SL_q(2,\mathbb{R})$.

Let $C_0^\infty(\mathbb{R}^2)$ be the space of all infinitely differentiable functions with finite support in $\mathbb{R}^2$. The linear functional on $C_0^\infty(\mathbb{R}^2)$:

$$\mathcal{I}_c(f) = \iint_{-\infty}^{\infty} dz_+ dz_- f(z_+, z_-). \tag{7.12}$$

where $f \in C_0^\infty(\mathbb{R}^2)$, is clearly the invariant integral on the translation group satisfying

$$(\mathcal{I}_c \otimes id)(\xi_c \diamond \xi_c)(f) = \mathcal{I}_c(f), \qquad (id \otimes \mathcal{I}_c)(\xi_c \diamond \xi_c)(f) = \mathcal{I}_c(f). \tag{7.13}$$

Let $A_0(SL_q(2,\mathbb{R}))$ be the subspace of $A(SL_q(2,\mathbb{R}))$ defined as

$$A_0(SL_q(2,\mathbb{R})) = C_0^\infty(\mathbb{R}^2) \times A(SL_q(2,\mathbb{R}|p)) \tag{7.14}$$

and $\mathcal{I}_w$ be the linear functional acting on it as

$$\mathcal{I}_w(F) = \sum_n \mathcal{I}_p(P_n)\mathcal{I}_c(f_n), \tag{7.15}$$

where $F = \sum_n P_n f_n$ and $f_n \in C_0^\infty(\mathbb{R}^2),\ P_n \in A(SL_q(2,\mathbb{R}|p))$. Let us prove that $\mathcal{I}_w$ is the invariant integral on $A_0(SL_q(2,\mathbb{R}))$. On an element $G = Pf$ we have

$$\mathcal{I}_w \diamond G = (id \otimes \mathcal{I}_w)\Delta(P)\Delta(f). \tag{7.16}$$

One can observe from (3.14) that any function $f(z)$ evaluated at $z = z_0$ can be written as

$$f(z)|_{z_0} = \xi_c \diamond \xi_c(f(z)).$$

Hence, (7.16) yields

$$\mathcal{I}_w \diamond G = (id \otimes \mathcal{I}_w)\left\{\Delta(P)\left[\xi_c \diamond \xi_c(f) + c_+\xi_c \diamond \xi_c(f'_+) + c_-\xi_c \diamond \xi_c(f'_-) + c_+c_-\xi_c \diamond \xi_c(f''_{+-})\right]\right\}.$$

by making use of (3.10). Moreover, the properties of the invariant integrals (7.3), (7.12) and (7.15) permits us to write

$$\mathcal{I}_w \diamond G = \mathcal{I}_p(P)\mathcal{I}_c(f) + (id \otimes \mathcal{I}_p)\left\{\Delta(P)\left[c_+\mathcal{I}_c(f'_+) + c_-\mathcal{I}_c(f'_-) + c_+c_-, \mathcal{I}_c(f''_{+-})\right]\right\}. \tag{7.17}$$

Because $f \in C_0^\infty(\mathbb{R}^2)$, we have

$$\mathcal{I}_c(\frac{df}{dz_\pm}) = \mathcal{I}_c(\frac{d^2 f}{dz_+ dz_-}) = 0. \tag{7.18}$$

Hence,

$$\mathcal{I}_w \diamond G = \mathcal{I}_p(P)\mathcal{I}_c(f) = \mathcal{I}_w(G), \tag{7.19}$$

which together with the linearity of the functional $\mathcal{I}_w$ implies

$$\mathcal{I}_w \diamond F = \mathcal{I}_w(F) \quad \text{for any} \quad F \in A_0(SL_q(2,\mathbb{R})). \tag{7.20}$$

Right invariance condition can be proved similarly. Therefore, $\mathcal{I}_w$ is the invariant integral on the quantum group $SL_q(2,\mathbb{R})$ at roots of unity.

Let us introduce the bilinear form

$$(F,G)_w = \mathcal{I}_w(FG^*), \tag{7.21}$$

where $F, G \in A_0(SL_q(2,\mathbb{R}))$, which is Hermitian because

$$\mathcal{I}_w(F^*) = \overline{\mathcal{I}_w(F)}. \tag{7.22}$$

Consider the subspace of $A_0(SL_q(2,\mathbb{R}))$

$$A_0(H_q^{(1,1)}) = C_0^\infty(\mathbb{R}^2) \times A(\mathcal{C}_q^{(1,1)}), \tag{7.23}$$

whose arbitrary element X can be written as

$$X = \sum_{nm} f_{nm}^+ e_{nm}^+ + \sum_{nm} f_{nm}^- e_{nm}^-, \tag{7.24}$$

where $e_{nm}^\pm$ are given by (3.19) in the domain (3.20). We then have

$$(X,X)_w = \sum_{nm} \mathcal{I}_c(f_{nm}^+ \overline{f_{nm}^+}) - \sum_{nm} \mathcal{I}_c(f_{nm}^- \overline{f_{nm}^-}). \tag{7.25}$$

Thus, $A_0(H_q^{(1,1)})$ endowed with the Hermitian form (7.21) is a pseudo–Euclidean space.

The comultiplication

$$\Delta : A_0(H_q^{(1,1)}) \to A_0(SL_q(2,\mathbb{R})) \otimes A_0(H_q^{(1,1)}),$$

defines the left quasi–regular representation of $SL_q(2,\mathbb{R})$ in $A_0(H_q^{(1,1)})$. Let us extend the Hermitian form $(\ ,\ )_w$ to $\{\ ,\ \}_w$ by setting

$$\{F \otimes X, G \otimes Y\}_w \equiv FG^*(X,Y)_w,$$

where $F,\ G \in A_0(SL_q(2,\mathbb{R}))$ and $X,\ Y \in A_0(H_q^{(1,1)})$. We have

$$\{\Delta(X), \Delta(Y)\}_w = 1_A (X,Y)_w, \tag{7.26}$$

which implies that the left quasi–regular representation is pseudo–unitary.

For any $\phi \in U_q(sl(2,\mathbb{R}))$ and $F \in A_0(SL_q(2,\mathbb{R}))$ the duality brackets satisfy the property

$$\overline{\langle \phi^*, F \rangle} = \langle \phi, (S(F))^* \rangle,$$

which together with the pseudo–unitarity condition (7.26) implies

$$(\mathcal{R}(\phi)X, Y)_w = (X, \mathcal{R}(\phi^*)Y)_w.$$

Thus, the antihomomorphism $\mathcal{R} : U_q(sl(2,\mathbb{R})) \to \mathrm{Lin} A_0(H_q^{(1,1)})$ given in Section 6 defines the $*$–representation of the quantum algebra in the pseudo–Euclidean space $A_0(H_q^{(1,1)})$.

Note that the matrix elements of the pseudo–unitary irreducible representations (5.10), (5.11) satisfy the orthogonality condition

$$(D^{\lambda}_{n0}, D^{\lambda'}_{m0})_w = \delta(\lambda_+ - \lambda'_+)\delta(\lambda_- - \lambda'_-) N_i \delta_{n+m,0(\mathrm{mod}\ p)},$$

where N_n are some normalization constants.

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